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A unified approach to finite-time hyperbolicity which extends finite-time Lyapunov exponents

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ABSTRACT

A hyperbolicity notion for linear differential equations $\dot{x} = A(t)x$, $t \in [t_-, t_+]$, is defined which unifies different existing notions like finite-time Lyapunov exponents (Haller, 2001, [13], Shadden et al., 2005, [24]), uniform or M-hyperbolicity (Haller, 2001, [13], Berger et al., 2009, [6]) and $(t_-, (t_+ - t_-))$-dichotomy (Rasmussen, 2010, [21]). Its relation to the dichotomy spectrum (Sacker and Sell, 1978, [23], Siegmund, 2002, [26]), D-hyperbolicity (Berger et al., 2009, [6]) and real parts of the eigenvalues (in case A is constant) is described. We prove a spectral theorem and provide an approximation result for the spectral intervals.

1. Introduction

Consider a linear nonautonomous differential equation on an interval $I \subseteq \mathbb{R}$

$$\dot{x}(t) = A(t)x(t), \quad t \in I, \tag{1}$$

where $A : I \to \mathbb{R}^{d\times d}$ is a continuous matrix-valued function. Let $\Phi : I \times I \to \mathbb{R}^{d\times d}$ denote the *evolution operator* of (1), i.e. $\Phi(\cdot, s)\xi$ is the solution of (1) such that $\Phi(s, s)\xi = \xi$ for any $s \in I$ and $\xi \in \mathbb{R}^d$. A projection-valued function $P : I \to \mathbb{R}^{d\times d}$ is called an *invariant projector* of (1) if for all $t, s \in I$

$$\Phi(t, s)P(s) = P(t)\Phi(t, s).$$

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Recall that for $I=\mathbb{R}$, (1) is termed *hyperbolic* if it admits an *exponential dichotomy* (*ED*) [26], i.e. there exist an invariant projector $P: I \to \mathbb{R}^{d\times d}$ and constants $\alpha,\beta>0$, $K\geqslant 1$ such that for all $t,s\in I$ and $\xi\in\mathbb{R}^d$,

$$\left\|\Phi(t,s)P(s)\xi\right\| \leqslant K\mathrm{e}^{-\alpha(t-s)}\|\xi\| \quad \text{for all } t\geqslant s,$$
$$\left\|\Phi(t,s)\big(\mathrm{id}-P(s)\big)\xi\right\| \leqslant K\mathrm{e}^{\beta(t-s)}\|\xi\| \quad \text{for all } t\leqslant s.$$

The *dichotomy* (*or Sacker–Sell*) *spectrum* of (1) is defined by

$$\Sigma_{\mathrm{dich}}(A) := \big\{\gamma\in\mathbb{R}\colon\ \dot{x}=\big[A(t)-\gamma\,\mathrm{id}\big]x \text{ does not admit an ED}\big\}.$$

For linear skew-product flows the dichotomy spectrum was established in [23]. For system (1) with $I=\mathbb{R}$ we have

Theorem 1. *(See [26].) Assume that $\|\Phi(t,s)\|\leqslant K\mathrm{e}^{\alpha|t-s|}$ for some $K\geqslant 1$, $\alpha>0$. Then the dichotomy spectrum $\Sigma_{\mathrm{dich}}(A)$ of* (1) *is the union of at most d non-empty disjoint compact* (*possibly one-point*) *intervals, called* spectral intervals, *i.e.*

$$\Sigma_{\mathrm{dich}}(A) = [a_1,b_1]\cup\cdots\cup[a_n,b_n],$$

where $n\in\{1,\dots,d\}$. Associated with the spectral intervals are uniquely determined sets $\mathcal{W}_1,\dots,\mathcal{W}_n\subset\mathbb{R}\times\mathbb{R}^d$ consisting of solutions of (1) (*linear integral manifolds*) *satisfying*

$$\mathcal{W}_1\oplus\cdots\oplus\mathcal{W}_n=\mathbb{R}\times\mathbb{R}^d.$$

For system (1) on a compact (finite-time) interval $I=[t_-,t_+]$ there have been introduced several hyperbolicity notions. In Haller [13], analytic criteria for the existence of finite-time uniformly attracting and repelling material surfaces were provided. Motivated by this work the notion of M-hyperbolicity was introduced in Berger et al. [6]. M-hyperbolicity is based on monotonic growth and decay of solutions. Another finite-time hyperbolicity notion is based on the EPH-partition in Haller [12,14], see also Duc and Siegmund [10,9], which was extended in Berger et al. [4] and called dynamic partition. To distinguish the different hyperbolicity notions, Berger et al. [4,5,3] later named a solution D-hyperbolic, if it is hyperbolic in the sense of the dynamic partition. Rasmussen [21] introduced (t_-,T)-dichotomies for $T=t_+-t_-$ which define a type of hyperbolicity for $\dot{x}=[A(t)-\gamma\,\mathrm{id}]x$ for certain $\gamma\in\mathbb{R}$ which turns out to be closely related to finite-time Lyapunov exponents (cp. Remark 3 and Theorem 24).

The paper is organized as follows: In Section 2 we define the new hyperbolicity notion (Definition 2) and observe in Remark 3 some of its basic properties. For reference we also recall D-hyperbolicity. Section 3 is split into three parts. The first subsection is devoted to a spectral theorem based on finite-time hyperbolicity. The main result (Theorem 10) shows – similar to the statement in Theorem 1 – that the spectrum is the union of at most d disjoint compact intervals. It generalizes results in [6, Theorem 17], [21, Theorem 4.13] and [8, Theorem 12]. In the second subsection the relation between the spectrum based on finite-time hyperbolicity and the dichotomy spectrum (Theorem 11), as well as the notion of D-hyperbolicity (Theorem 15), is clarified. The constant coefficient case $\dot{x}=Ax$ is treated as a special case (Corollary 13). The third subsection of Section 3 addresses the issue of approximation of spectral intervals. Theorem 20 provides core ingredients for an algorithm to numerically compute the finite-time spectrum. In Section 4 we discuss the established concept of finite-time Lyapunov exponents as a special case of finite-time spectrum (Theorem 24).

We introduce some notions which are used in the paper. For a matrix $M\in\mathbb{R}^{d\times d}$ let $\delta_1(M),\dots,\delta_d(M)$ denote its singular values.

For a compact set J let $|J| \in \mathbb{N}_0 \cup \{\infty\}$ denote its cardinality and

$$\rho(J) := \inf_{x,y \in J,\, x \neq y} |x-y|.$$

As usual, the Hausdorff distance between two compact sets $I, J \subset \mathbb{R}$ is defined by

$$d_H(I,J) := \max\bigl\{d(I,J), d(J,I)\bigr\}$$

with $d(I,J) := \sup_{x\in I} \inf_{y\in J} |x-y|$.

For $j,d \in \mathbb{N}$, $j \leqslant d$, let $G_{j,d}$ denote the Grassmannian consisting of all subspaces in $\mathbb{R}^d$ of dimension j. We equip $G_{j,d}$ with the following metric

$$d_G : G_{j,d} \times G_{j,d} \to \mathbb{R}_{\geqslant 0}, \quad d_G(X,Y) := \|\pi_X - \pi_Y\|,$$

where π_X, π_Y denote the orthogonal projections onto X and Y, respectively. $(G_{j,d}, d_G)$ is a compact metric space (see e.g. [7]). We denote by $G_d := \bigcup_{j=1}^d G_{j,d}$ the set of all non-trivial subspaces of $\mathbb{R}^d$. Note that in [4,8] D-hyperbolicity is studied for $J = I = [t_-, t_+]$.

2. Finite-time hyperbolicity

We define a new finite-time hyperbolicity notion for system (1) w.r.t. a compact subset $J \subset I$.

Definition 2 *(Finite-time hyperbolicity).* Let $\mathbb{R}^d$ be endowed with an arbitrary norm $\|\cdot\|$ and J a compact subset of I with $|J| \geqslant 2$. System (1) is called *finite-time hyperbolic* with respect to the norm $\|\cdot\|$ and the set J if there exist an invariant projector P and constants $\alpha, \beta > 0$ such that for all $t,s \in J$ with $t \geqslant s$

$$\bigl\|\Phi(t,s)\xi\bigr\| \leqslant \mathrm{e}^{-\alpha(t-s)}\|\xi\| \quad \text{for all } \xi \in \operatorname{im} P(s),$$
$$\bigl\|\Phi(t,s)\xi\bigr\| \geqslant \mathrm{e}^{\beta(t-s)}\|\xi\| \quad \text{for all } \xi \in \ker P(s).$$

Remark 3. (i) Let J and $\widetilde{J}$ be two compact subsets of I such that $\widetilde{J} \subset J$. Assume that system (1) is finite-time hyperbolic with respect to J. Then (1) is also finite-time hyperbolic with respect to $\widetilde{J}$.

(ii) Suppose that $J = \{t_1, \ldots, t_k\} \subset I$, where $t_1 < t_2 < \cdots < t_k$. Then system (1) is finite-time hyperbolic with respect to J if and only if there exists an invariant projector P such that for each $i \in \{1, \ldots, k-1\}$ we have

$$\bigl\|\Phi(t_{i+1},t_1)\xi\bigr\| < \bigl\|\Phi(t_i,t_1)\xi\bigr\| \quad \text{for all } \xi \in \operatorname{im} P(t_1) \setminus \{0\},$$
$$\bigl\|\Phi(t_{i+1},t_1)\xi\bigr\| > \bigl\|\Phi(t_i,t_1)\xi\bigr\| \quad \text{for all } \xi \in \ker P(t_1) \setminus \{0\}.$$

For $k=2$ we get as a special case the definition of nonhyperbolic $(t_1, t_2 - t_1)$-dichotomy as in [21].

(iii) In Section 4 we introduce finite-time Lyapunov exponents for system (1) given on $I = [t_-, t_+]$. All finite-time Lyapunov exponents of system (1) are non-zero if and only if (1) is finite-time hyperbolic with respect to $J = \{t_-, t_+\}$ (see Theorem 24).

To conclude this section, we recall the notion of D-hyperbolicity which is a sufficient condition for M-hyperbolicity [8, Corollary 22] (for more details see also [4,8]). For this purpose we consider system (1) and assume additionally that $A : I \to \mathbb{R}^{d\times d}$ is differentiable and that the norm in $\mathbb{R}^d$ is induced by a symmetric positive definite matrix $\Gamma \in \mathbb{R}^{d\times d}$, i.e. $\|x\|_\Gamma = \sqrt{\langle x, \Gamma x\rangle}$ for $x \in \mathbb{R}^d$. The symmetric matrix

$$S_\Gamma(t) := \frac{1}{2}\left[\Gamma A(t) + A(t)^\top \Gamma\right] \tag{2}$$

is called the *Γ-strain tensor* of Eq. (1) and describes for an arbitrary solution $\xi : I \to \mathbb{R}^d$ of (1) the instantaneous change of $\frac{1}{2}\|\xi\|_\Gamma^2$ by

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\xi(t)\right\|_\Gamma^2 = \left\langle \xi(t), S_\Gamma(t)\xi(t)\right\rangle.$$

Thus the Γ-strain tensor describes growth and decay of solutions ξ of (1) with respect to the $\|\cdot\|_\Gamma$-norm. Clearly, all non-trivial solutions of (1) are strictly decreasing or increasing with respect to the $\|\cdot\|_\Gamma$-norm if $S_\Gamma(t)$ is negative or positive definite, respectively.

The set

$$Z_\Gamma(t) := \left\{\xi \in \mathbb{R}^d \colon \left\langle \xi, S_\Gamma(t)\xi\right\rangle = 0\right\}$$

is called the *zero Γ-strain set* of Eq. (1). It is a non-trivial cone if $S_\Gamma(t)$ has both positive and negative eigenvalues. To describe the dynamical behavior of solutions of (1) starting in the set $Z_\Gamma(t)$, one defines the so-called *Γ-strain acceleration tensor*

$$M_\Gamma(t) := \dot{S}_\Gamma(t) + S_\Gamma(t)A(t) + A(t)^\top S_\Gamma(t),$$

which satisfies $\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\|\xi(t)\|_\Gamma^2 = \langle \xi(t), M_\Gamma(t)\xi(t)\rangle$. The restriction of the quadratic form $\xi \mapsto \langle \xi, M_\Gamma(t)\xi\rangle$ to $Z_\Gamma(t)$ is denoted by $M_{Z_\Gamma}(t)$. Based on [4, Definition 2.4] we say that for a compact set $J \subset I$

> system (1) is *D-hyperbolic* on J if $S_\Gamma(t)$ is indefinite and non-degenerate, and $M_{Z_\Gamma}(t)$ is positive definite for all $t \in J$.

Similarly, (1) is called *D-quasi-hyperbolic* or *D-elliptic* if $M_{Z_\Gamma}(t)$ is negative definite or indefinite, instead of positive definite, respectively. System (1) is called *D-attracting* or *D-repelling* on J if $S_\Gamma(t)$ is negative or positive definite, respectively, for all $t \in J$.

3. Finite-time spectrum

For $\gamma \in \mathbb{R}$, we consider the shifted equation

$$\dot{x}(t) = \left[A(t) - \gamma\,\mathrm{id}\right]x(t), \tag{3}$$

and denote its evolution operator by $\Phi_\gamma : I \times I \to \mathbb{R}^{d\times d}$. Note that $\Phi_\gamma(t,s) = \mathrm{e}^{-\gamma(t-s)}\Phi(t,s)$ for all $t, s \in I$.

Definition 4 *(Finite-time spectrum).* Let $\mathbb{R}^d$ be endowed with an arbitrary norm $\|\cdot\|$ and J a compact subset of $[t_-, t_+]$. The *finite-time spectrum* of (1) with respect to J and $\|\cdot\|$ is the set

$$\Sigma_J(A) = \left\{\gamma \in \mathbb{R}\colon \text{Eq. (3) is not finite-time hyperbolic w.r.t. } J \text{ and } \|\cdot\|\right\},$$

and the complement $\rho_J(A) = \mathbb{R} \setminus \Sigma_J(A)$ is called *finite-time resolvent set* of (1) with respect to J and $\|\cdot\|$.

In the next part of this section, we establish a spectral theorem which characterizes the finite-time spectrum by extremal growth rates. Later, we formulate some asymptotic and instantaneous aspects of the finite-time spectrum. The remaining part is devoted to present an approximation result for finite-time spectral intervals.

3.1. A spectral theorem for finite-time differential equations

Definition 5 *(Growth rates).* Let $X \in G_d$ and J a compact subset of I. Set $t_1 := \min J$. We call

$$\underline{\lambda}(J, X) := \sup\left\{\alpha \in \mathbb{R} \;\middle|\; \forall \xi \in X : t \mapsto \mathrm{e}^{-\alpha t} \left\| \Phi(t, t_1)\xi \right\| \text{ is increasing on } J\right\},$$

and

$$\bar{\lambda}(J, X) := \inf\left\{\alpha \in \mathbb{R} \;\middle|\; \forall \xi \in X : t \mapsto \mathrm{e}^{-\alpha t} \left\| \Phi(t, t_1)\xi \right\| \text{ is decreasing on } J\right\},$$

respectively, the *lower* and *upper growth rate of* X (or the integral manifold induced by X) with respect to J. We extend the definition by $\underline{\lambda}(J, \{0\}) = +\infty$ and $\bar{\lambda}(J, \{0\}) = -\infty$ in a natural way.

In the case that J is finite, explicit expressions for the lower and upper growth rates of an $X \in G_d$ are given as follows.

Remark 6. Let $X \in G_d$. Suppose that $J = \{t_1, \dots, t_k\} \subset I$, where $t_1 < t_2 < \dots < t_k$. Then

$$\underline{\lambda}(J, X) = \inf_{\xi \in X \cap \mathcal{S}^{d-1}} \min_{i \in \{1,\dots,k-1\}} \frac{1}{t_{i+1} - t_i} \log \frac{\|\Phi(t_{i+1}, t_1)\xi\|}{\|\Phi(t_i, t_1)\xi\|},$$

$$\bar{\lambda}(J, X) = \sup_{\xi \in X \cap \mathcal{S}^{d-1}} \max_{i \in \{1,\dots,k-1\}} \frac{1}{t_{i+1} - t_i} \log \frac{\|\Phi(t_{i+1}, t_1)\xi\|}{\|\Phi(t_i, t_1)\xi\|}.$$

We now introduce the notion of extremal growth rates which play an important role in determining the spectrum.

Definition 7 *(Extremal k-dimensional growth rates).* For $k \in \{0, \dots, d\}$, the numbers $\underline{\lambda}^{(k)}(J)$ and $\bar{\lambda}^{(k)}(J)$ defined by

$$\underline{\lambda}^{(k)}(J) := \sup_{X \in G_{k,d}} \underline{\lambda}(J, X), \quad \text{and} \quad \bar{\lambda}^{(k)}(J) := \inf_{X \in G_{k,d}} \bar{\lambda}(J, X), \tag{4}$$

are called, respectively, the *maximal k-dimensional lower growth rate* and the *minimal k-dimensional upper growth rate* of (1) with respect to the compact set $J \subset I$. Note that $\underline{\lambda}^{(0)}(J) = +\infty$ and $\bar{\lambda}^{(0)}(J) = -\infty$, according to Definition 5.

The question if there exist subspaces realizing the maximal lower and minimal upper growth rates arises naturally. In order to answer this question we introduce the following functions: For each $k \in \{0, 1, \dots, d\}$, the *lower and upper growth rate functions* $\underline{\lambda}^{(k)}(J, \cdot), \bar{\lambda}^{(k)}(J, \cdot) : G_{k,d} \to \mathbb{R}$ are defined by

$$\underline{\lambda}^{(k)}(J, X) := \underline{\lambda}(J, X), \qquad \bar{\lambda}^{(k)}(J, X) := \bar{\lambda}(J, X).$$

Lemma 8. *For each $k \in \{0, \dots, d\}$ and compact set $J \subset I$, the lower growth rate function $\underline{\lambda}^{(k)}(J, \cdot)$ is upper semi-continuous and the upper growth rate function $\bar{\lambda}^{(k)}(J, \cdot)$ is lower semi-continuous.*

Proof. Analogous to the proof of [8, Lemma 7]. □

Remark 9. (i) Lemma 8 in combination with the compactness of the metric space $(G_{k,d}, d_G)$ implies that there exists $X \in G_{k,d}$ such that $\underline{\lambda}(J, X) = \underline{\lambda}^{(k)}(J)$. We also say that X realizes the maximal lower growth rate $\underline{\lambda}^{(k)}(J)$. Similarly, there exists $Y \in G_{k,d}$ realizing the minimal upper growth rate $\bar{\lambda}^{(k)}(J)$.

(ii) The subspaces realizing the maximal and minimal growth rates need not be unique, see e.g. [6, Example 27].

The dichotomy spectrum of system (1) for $I = \mathbb{R}$ is the union of at most d disjoint, non-empty intervals and in case the solutions are exponentially bounded those intervals are compact (Theorem 1). The following theorem states that for system (1) on an arbitrary interval $I \subset \mathbb{R}$ the finite-time spectrum with respect to an arbitrary norm and a compact set $J \subset I$ has the same property. Moreover, based on the maximal lower and minimal upper growth rates an explicit expression for Σ_J is given. This result generalizes [6, Theorem 17], [21, Theorem 4.13] and [8, Theorem 10].

Theorem 10 *(Finite-time spectral intervals). Let J be a compact subset of I and $\mathbb{R}^d$ be endowed with an arbitrary norm. The spectrum $\Sigma_J(A)$ of* (1) *is the union of at most d disjoint, non-empty intervals* (*called spectral intervals*)*, i.e.*

$$\Sigma_J(A) = \bigcup_{\ell=1}^{n} [a_\ell, b_\ell],$$

where $n \in \{1, \dots, d\}$ and $-\infty < a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_n \leqslant b_n < +\infty$. Moreover, let $\Lambda := \{i_0, \dots, i_n\}$, $0 = i_0 < \cdots < i_n = d$, be the set of all indices $j \in \{0, \dots, d\}$ satisfying $\bar{\lambda}^{(j)}(J) < \underline{\lambda}^{(d-j)}(J)$. Then, for each $k \in \{1, \dots, n\}$

$$a_k = \underline{\lambda}^{(d-i_{k-1})}(J), \qquad b_k = \bar{\lambda}^{(i_k)}(J).$$

Proof. Set $t_1 := \min J$. By (4) and Definition 5, the sequence of minimal upper growth rates $(\bar{\lambda}^{(k)}(J))_{k\in\{0,\dots,d\}}$ is increasing and the sequence of maximal lower growth rates $(\underline{\lambda}^{(k)}(J))_{k\in\{0,\dots,d\}}$ is decreasing. Furthermore, for each $k \in \{0, \dots, d\}$ one has $\underline{\lambda}^{(d-k+1)}(J) \leqslant \bar{\lambda}^{(k)}(J)$. As a consequence, we get for $\ell \neq k$

$$\big(\bar{\lambda}^{(i_\ell)}(J), \underline{\lambda}^{(d-i_\ell)}(J)\big) \cap \big(\bar{\lambda}^{(i_k)}(J), \underline{\lambda}^{(d-i_k)}(J)\big) = \emptyset. \tag{5}$$

We now show that

$$\rho_J(A) = \bigcup_{k=0}^{n} \big(\bar{\lambda}^{(i_k)}(J), \underline{\lambda}^{(d-i_k)}(J)\big). \tag{6}$$

For this purpose, let $\gamma \in (\bar{\lambda}^{(i_k)}(J), \underline{\lambda}^{(d-i_k)}(J))$ for some $k \in \{0, 1, \dots, n\}$. Consider the corresponding shifted equation

$$\dot{x}(t) = \big[A(t) - \gamma\,\mathrm{id}\big]x(t). \tag{7}$$

By (4), there exist subspaces W_s and W_u of dimension i_k and $d - i_k$, respectively, and $\alpha > 0$ such that

$$\bar{\lambda}(J, W_s) \leqslant \gamma - \alpha, \qquad \underline{\lambda}(J, W_u) \geqslant \gamma + \alpha.$$

Therefore, the function $t \mapsto \mathrm{e}^{-(\gamma-\alpha)t}\|\Phi(t,t_1)\xi\| = \mathrm{e}^{\alpha t}\|\Phi_\gamma(t,t_1)\xi\|\mathrm{e}^{-\gamma t_1}$ is decreasing on J for all $\xi \in W_s$. Similarly, the function

$$t \mapsto \mathrm{e}^{-(\gamma+\alpha)t}\big\|\Phi(t,t_1)\xi\big\| = \mathrm{e}^{-\alpha t}\big\|\Phi_\gamma(t,t_1)\xi\big\|\mathrm{e}^{-\gamma t_1}$$

is increasing on J for all $\xi \in W_u$. Thus, system (7) is hyperbolic with respect to the set J. Conversely, let $\gamma \in \rho_J(A)$ and suppose that system (7) is hyperbolic with respect to the set J with an invariant projector $P: I \to \mathbb{R}^{d\times d}$. Then there exists $\alpha > 0$ such that for all $t, s \in J$ with $t \geqslant s$ and all $\xi \in \ker P(t_1)$

$$\left\| \Phi_\gamma(t, t_1)\xi \right\| \geqslant e^{\alpha(t-s)} \left\| \Phi_\gamma(s, t_1)\xi \right\|.$$

Consequently, $\underline{\lambda}^{(d-k)}(J) \geqslant \underline{\lambda}(J, \ker P(t_1)) \geqslant \alpha + \gamma$, where $k := \dim \operatorname{im} P(t_1)$. Similarly, we get $\bar{\lambda}^{(k)}(J) \leqslant \gamma - \alpha$. Thus $\gamma \in (\bar{\lambda}^{(k)}(J), \underline{\lambda}^{(d-k)}(J))$ and hence (6) is proved. This together with (5) implies that

$$\Sigma_J(A) = \bigcup_{\ell=1}^{n} \left[\underline{\lambda}^{(d-i_{\ell-1})}(J), \bar{\lambda}^{(i_\ell)}(J)\right],$$

which completes the proof. □

3.2. Relation to dichotomy spectrum and D-hyperbolicity

In this subsection we study the dependence of $\Sigma_J(A)$ on $J \subset I$. In the first case we investigate the *asymptotic* relation of $\Sigma_J(A)$ to the dichotomy spectrum $\Sigma_{\text{dich}}(A)$ on $I = \mathbb{R}$ by increasing $\rho(J)$ to infinity, in the second case we decrease $\rho(J)$ to zero and get a relation of $\Sigma_J(A)$ to the *instantaneous* growth rates of solutions described by the Γ-strain tensor $S_\Gamma(t_0)$ which is used for the definition of D-hyperbolicity.

Theorem 11 *(Relation to dichotomy spectrum). Suppose that system* (1) *is defined on the whole real line* $\mathbb{R}$ *and its evolution operator* $\Phi(\cdot,\cdot)$ *has bounded growth, i.e. there exist* $K \geqslant 1$ *and* $\alpha > 0$ *such that* $\|\Phi(t,s)\| \leqslant Ke^{\alpha|t-s|}$ *for any* $t, s \in \mathbb{R}$. *Let* $(J_m)_{m\in\mathbb{N}}$ *be a sequence of finite sets, i.e.* $|J_m| < \infty$ *for each* $m \in \mathbb{N}$, *such that* $\lim_{m\to\infty} \rho(J_m) = \infty$. *Then*

$$\lim_{m\to\infty} d\big(\Sigma_{J_m}(A), \Sigma_{\text{dich}}(A)\big) = 0, \tag{8}$$

where $\Sigma_{\text{dich}}(A)$ *is the dichotomy spectrum of* (1) *on* $\mathbb{R}$. *Furthermore, if* $\Sigma_{\text{dich}}(A)$ *is a discrete set then*

$$\lim_{m\to\infty} d_H\big(\Sigma_{J_m}(A), \Sigma_{\text{dich}}(A)\big) = 0.$$

Proof. According to Theorem 1, the dichotomy spectrum of (1) on $\mathbb{R}$ has the following form

$$\Sigma_{\text{dich}}(A) = [a_1, b_1] \cup \cdots \cup [a_n, b_n],$$

where $n \in \{1, \dots, d\}$ and $a_1 \leqslant b_1 < \cdots < a_n \leqslant b_n$. Choose $\varepsilon > 0$ such that $\varepsilon \leqslant \frac{a_{k+1}-b_k}{2}$ for each $k \in \{1, \dots, n-1\}$. Then there exist integral manifolds $\mathcal{W}_1, \dots, \mathcal{W}_n \subset \mathbb{R} \times \mathbb{R}^d$ with $\bigoplus_{\ell=1}^n \mathcal{W}_\ell = \mathbb{R} \times \mathbb{R}^d$ and a constant $K \geqslant 1$ such that for all $k \in \{1, \dots, n\}$ and $t \geqslant s$

$$\left\| \Phi(t,s)\xi \right\| \geqslant \frac{1}{K} e^{(a_k - \frac{\varepsilon}{4})(t-s)} \|\xi\| \quad \text{for all } \xi \in \bigoplus_{\ell=k}^{n} \mathcal{W}_\ell(s), \tag{9}$$

$$\left\| \Phi(t,s)\xi \right\| \leqslant K e^{(b_k + \frac{\varepsilon}{4})(t-s)} \|\xi\| \quad \text{for all } \xi \in \bigoplus_{\ell=1}^{k} \mathcal{W}_\ell(s). \tag{10}$$

For $\ell \in \{1, \dots, n\}$ let $d_\ell := \dim \mathcal{W}_\ell$ and $i_k := \sum_{\ell=1}^k d_\ell$, then $i_n = \sum_{\ell=1}^n d_\ell = d$. Since $\lim_{m\to\infty} \rho(J_m) = \infty$ there exists $N(\varepsilon) \in \mathbb{N}$ such that $|\frac{\log K}{\rho(J_m)}| \leqslant \frac{\varepsilon}{4}$ for all $m \geqslant N(\varepsilon)$. Choose $m \in \mathbb{N}$, $m > N(\varepsilon)$, and let

$J_m = \{t_1, \dots, t_q\}$ with $t_1 < \cdots < t_q$. By the nesting property of the growth rates, shown in the first part of the proof of Theorem 10, the assertion is proved if we can show that for any $k \in \{1, \dots, n\}$ we have

$$a_k - \frac{\varepsilon}{2} \leqslant \underline{\lambda}^{(d-i_{k-1})}(J_m) \quad \text{and} \quad \overline{\lambda}^{(i_k)}(J_m) \leqslant b_k + \frac{\varepsilon}{2}. \tag{11}$$

To this end, let $X = \bigoplus_{\ell=k}^{n} \mathcal{W}_\ell(t_1) \in G_{d-i_{k-1},d}$ and $\xi \in X \cap \mathcal{S}^{d-1}$. By (9) we have for all $p \in \{1, \dots, q-1\}$

$$\begin{aligned}\frac{1}{t_{p+1}-t_p} \log \frac{\|\Phi(t_{p+1}, t_1)\xi\|}{\|\Phi(t_p, t_1)\xi\|} &= \frac{1}{t_{p+1}-t_p} \log \frac{\|\Phi(t_{p+1}, t_p)\Phi(t_p, t_1)\xi\|}{\|\Phi(t_p, t_1)\xi\|} \\ &\geqslant \frac{1}{t_{p+1}-t_p} \log \frac{e^{(a_k - \frac{\varepsilon}{4})(t_{p+1}-t_p)}}{K} \\ &\geqslant a_k - \frac{\varepsilon}{4} - \frac{\log K}{\rho(J_m)} \\ &\geqslant a_k - \frac{\varepsilon}{2}.\end{aligned}$$

By virtue of Remark 6 and Definition 7 we get that

$$a_k - \frac{\varepsilon}{2} \leqslant \underline{\lambda}(X, J_m) \leqslant \underline{\lambda}^{(d-i_{k-1})}(J_m).$$

Similarly, we have $\overline{\lambda}^{(i_k)} \leqslant b_k + \frac{\varepsilon}{2}$ for $k \in \{1, 2, \dots, n\}$ and (11) is proved. Combining (11) and Theorem 10, we obtain

$$d\big(\Sigma_{J_m}(A), \Sigma_{\mathrm{dich}}(A)\big) \leqslant \frac{\varepsilon}{2},$$

which proves (8). In the special case that $\Sigma_{\mathrm{dich}}(A)$ is a discrete set, we have $a_i = b_i$ for $i = 1, \dots, n$. Hence, when we replace the semi-distance d by the Hausdorff distance d_H in (8) then this statement remains true. The proof is complete. □

The following scalar system provides an example for the fact that if we replace the semi-distance d by the Hausdorff distance in (8), then this statement is in general not true.

Example 12. Consider the following linear nonautonomous scalar differential equation

$$\dot{x}(t) = \left[\sin\big(\ln\big(1+t^2\big)\big) + \frac{2t^2}{1+t^2} \cos\big(\ln\big(1+t^2\big)\big)\right] x(t). \tag{12}$$

The associated evolution operator is given by

$$\Phi(t, s) = e^{t \sin(\ln(1+t^2)) - s \sin(\ln(1+s^2))} \quad \text{for all } t, s \in \mathbb{R}.$$

Then the dichotomy spectrum is given by $\Sigma_{\mathrm{dich}} = [-1, 1]$. Define $J_m := \{-m, m\}$ for each $m \in \mathbb{N}$. Clearly, $\lim_{m\to\infty} \rho(J_m) = \infty$. By virtue of Theorem 10, the finite-time spectrum Σ_{J_m} is a singleton set given by

$$\Sigma_{J_m} = \left\{\frac{1}{2m} \log \Phi(m, -m)\right\} = \big\{\sin\big(\ln\big(1+m^2\big)\big)\big\} \quad \text{for all } m \in \mathbb{N},$$

which proves that

$$d_H(\Sigma_{\mathrm{dich}}, \Sigma_{J_m}) \geqslant \frac{1}{2} \quad \text{for all } m \in \mathbb{N}.$$

In the special case of an autonomous system $\dot{x} = Ax$ with $A \in \mathbb{R}^{d\times d}$ the dichotomy spectrum consists of the real parts of the eigenvalues of A, i.e. $\Sigma_{\mathrm{dich}}(A) = \mathrm{Re}[\sigma(A)] := \{\mathrm{Re}(\lambda)\colon\ \lambda \in \sigma(A)\}$ where $\sigma(A)$ denotes the spectrum of A (see e.g. [23,26]). The finite-time spectrum of $\dot{x} = Ax$ w.r.t. a compact set $J \subset R$ does not necessarily coincide with $\mathrm{Re}[\sigma(A)]$. In [21], however, it is proved (cp. also Remark 3(ii) for the relation between the hyperbolicity notion suggested in [21] and ours) that for $t, T \in \mathbb{R}$ and the two-point set $J = \{t, T\}$

$$\lim_{T\to\infty} d_H\big(\Sigma_{\{t,T\}}(A), \mathrm{Re}\big[\sigma(A)\big]\big) = 0.$$

As a corollary to Theorem 11 we can extend this approximation result to the finite-time spectrum of $\dot{x} = Ax$.

Corollary 13 *(Relation to real parts of the eigenvalues). For $A \in \mathbb{R}^{d\times d}$ consider the autonomous system*

$$\dot{x}(t) = Ax(t).$$

Let $(J_m)_{m\in\mathbb{N}}$ be a sequence of finite sets such that $\lim_{m\to\infty} \rho(J_m) = \infty$. Then,

$$\lim_{m\to\infty} d_H\big(\Sigma_{J_m}(A), \mathrm{Re}\big[\sigma(A)\big]\big) = 0.$$

The remaining part of this subsection is devoted to relate $\Sigma_{J_m}(A)$ for $\rho(J) \to 0$ to the instantaneous growth rates of solutions described by the Γ-strain tensor (2) which is used for the definition of D-hyperbolicity. For this purpose, we need the following preparatory lemma which relates the evolution operator $\Phi(t,s)$ of (1) for a given $\|\cdot\|_\Gamma$-norm to its Γ-strain tensor. Recall that $\delta_k(M)$ denotes the k-th singular value of a matrix M, which is identical to the k-th eigenvalue in case M is symmetric.

Lemma 14. *For any $t_0 \in I$ the following statements hold*:

(i) *For any $\xi \in \mathbb{R}^d \setminus \{0\}$, we have*

$$\lim_{t,s\to t_0} \frac{1}{t-s} \log \frac{\|\Phi(t,s)\xi\|_\Gamma}{\|\xi\|_\Gamma} = \frac{\langle \xi, S_\Gamma(t_0)\xi\rangle}{\|\xi\|_\Gamma^2}.$$

(ii) *For any $k \in \{1,\dots,d\}$, we have*

$$\lim_{t,s\to t_0} \frac{1}{t-s} \log \delta_k\big(\Gamma^{-\frac{1}{2}}\Phi(t,s)^\top \Gamma \Phi(t,s)\Gamma^{-\frac{1}{2}}\big) = 2\delta_k\big(\Gamma^{-\frac{1}{2}} S_\Gamma(t_0)\Gamma^{-\frac{1}{2}}\big).$$

Proof. (i) Integrating Eq. (1) yields $\Phi(t,s) = \mathrm{id} + \int_s^t A(u)\Phi(u,s)\,\mathrm{d}u$, and one has $\Phi(t,s) = \mathrm{id} + (t-s)A(t_0) + \int_s^t A(u)\Phi(u,s) - A(t_0)\,\mathrm{d}u$, such that

$$\Phi(t,s) = \mathrm{id} + (t-s)A(t_0) + o(t-s) \quad \text{as } t, s \to t_0.$$

Consequently, we obtain that

$$\begin{aligned}\lim_{t,s\to t_0}\frac{1}{t-s}\log\frac{\|\Phi(t,s)\xi\|_\Gamma}{\|\xi\|_\Gamma} &= \lim_{t,s\to t_0}\frac{1}{2(t-s)}\log\frac{\langle\Phi(t,s)\xi,\Gamma\Phi(t,s)\xi\rangle}{\langle\xi,\Gamma\xi\rangle}\\ &= \lim_{t,s\to t_0}\frac{1}{2(t-s)}\log\left(1+2(t-s)\frac{\langle\xi,S_\Gamma(t_0)\xi\rangle}{\|\xi\|_\Gamma^2}+o(t-s)\right)\\ &= \frac{\langle\xi,S_\Gamma(t_0)\xi\rangle}{\|\xi\|_\Gamma^2}.\end{aligned}$$

(ii) Using the Courant–Fischer Min–max Theorem (see e.g. [11])

$$\begin{aligned}\delta_k\big(\Gamma^{-\frac12}S_\Gamma(t_0)\Gamma^{-\frac12}\big) &= \max_{X\in G_{k,d}}\min_{\xi\in X\setminus\{0\}}\frac{\langle\xi,\Gamma^{-\frac12}S_\Gamma(t_0)\Gamma^{-\frac12}\xi\rangle}{\langle\xi,\xi\rangle}\\ &= \max_{X\in G_{k,d}}\min_{\xi\in X\setminus\{0\}}\frac{\langle\xi,S_\Gamma(t_0)\xi\rangle}{\langle\xi,\Gamma\xi\rangle}.\end{aligned}$$

Let $X_k\in G_{k,d}$ such that

$$\delta_k\big(\Gamma^{-\frac12}S_\Gamma(t_0)\Gamma^{-\frac12}\big)=\min_{\xi\in X_k\setminus\{0\}}\frac{\langle\xi,S_\Gamma(t_0)\xi\rangle}{\langle\xi,\Gamma\xi\rangle}.$$

Similarly, using the Courant–Fischer Min–max Theorem we also have

$$\log\delta_k\big(\Gamma^{-\frac12}\Phi(t,s)^\top\Gamma\Phi(t,s)\Gamma^{-\frac12}\big)=\max_{X\in G_{k,d}}\min_{\xi\in X\setminus\{0\}}\log\frac{\|\Phi(t,s)\xi\|_\Gamma^2}{\|\xi\|_\Gamma^2}.$$

Consequently, we have

$$\log\delta_k\big(\Gamma^{-\frac12}\Phi(t,s)^\top\Gamma\Phi(t,s)\Gamma^{-\frac12}\big)=-\log\delta_k\big(\Gamma^{-\frac12}\Phi(s,t)^\top\Gamma\Phi(s,t)\Gamma^{-\frac12}\big)$$

and

$$\log\delta_k\big(\Gamma^{-\frac12}\Phi(t,s)^\top\Gamma\Phi(t,s)\Gamma^{-\frac12}\big)\geqslant 2\min_{\xi\in X_k\setminus\{0\}}\log\frac{\|\Phi(t,s)\xi\|_\Gamma}{\|\xi\|_\Gamma},$$

which together with part (i) implies that

$$\begin{aligned}&\liminf_{t,s\to t_0}\frac{1}{t-s}\log\delta_k\big(\Gamma^{-\frac12}\Phi(t,s)^\top\Gamma\Phi(t,s)\Gamma^{-\frac12}\big)\\ &\quad=\liminf_{t,s\to t_0,\,t>s}\frac{1}{t-s}\log\delta_k\big(\Gamma^{-\frac12}\Phi(t,s)^\top\Gamma\Phi(t,s)\Gamma^{-\frac12}\big)\\ &\quad\geqslant 2\liminf_{t,s\to t_0,\,t>s}\min_{\xi\in X_k\setminus\{0\}}\frac{1}{t-s}\log\frac{\|\Phi(t,s)\xi\|_\Gamma}{\|\xi\|_\Gamma}\\ &\quad\geqslant 2\min_{\xi\in X_k\setminus\{0\}}\liminf_{t,s\to t_0,\,t>s}\frac{1}{t-s}\log\frac{\|\Phi(t,s)\xi\|_\Gamma}{\|\xi\|_\Gamma}\\ &\quad\geqslant 2\delta_k\big(\Gamma^{-\frac12}S_\Gamma(t_0)\Gamma^{-\frac12}\big).\end{aligned}$$

On the other hand, using the Courant–Fischer Min–max Theorem we have

$$\delta_k\big(\Gamma^{-\frac{1}{2}} S_\Gamma(t_0)\Gamma^{-\frac{1}{2}}\big) = \min_{X\in G_{d-k,d}} \max_{\xi\in X\setminus\{0\}} \frac{\langle\xi, S_\Gamma(t_0)\xi\rangle}{\langle\xi,\Gamma\xi\rangle}.$$

Let $X_{d-k}\in G_{d-k,d}$ such that

$$\delta_k\big(\Gamma^{-\frac{1}{2}} S_\Gamma(t_0)\Gamma^{-\frac{1}{2}}\big) = \max_{\xi\in X_{d-k}\setminus\{0\}} \frac{\langle\xi, S_\Gamma(t_0)\xi\rangle}{\langle\xi,\Gamma\xi\rangle}.$$

In view of the Courant–Fischer Min–max Theorem and (i), we get

$$\begin{aligned}
&\limsup_{t,s\to t_0} \frac{1}{t-s}\log\delta_k\big(\Gamma^{-\frac{1}{2}}\Phi(t,s)^\top\Gamma\Phi(t,s)\Gamma^{-\frac{1}{2}}\big)\\
&\quad= \limsup_{t,s\to t_0,\, t>s} \frac{1}{t-s}\log\delta_k\big(\Gamma^{-\frac{1}{2}}\Phi(t,s)^\top\Gamma\Phi(t,s)\Gamma^{-\frac{1}{2}}\big)\\
&\quad\leqslant \limsup_{t,s\to t_0,\, t>s} \max_{\xi\in X_{d-k}\setminus\{0\}} \frac{1}{t-s}\log\frac{\langle\xi,\Phi(t,s)^\top\Gamma\Phi(t,s)\xi\rangle}{\langle\xi,\Gamma\xi\rangle}\\
&\quad\leqslant 2\max_{\xi\in X_{d-k}\setminus\{0\}} \limsup_{t,s\to t_0,\, t>s} \frac{1}{t-s}\log\frac{\|\Phi(t,s)\xi\|_\Gamma}{\|\xi\|_\Gamma}\\
&\quad= 2\delta_k\big(\Gamma^{-\frac{1}{2}} S_\Gamma(t_0)\Gamma^{-\frac{1}{2}}\big).
\end{aligned}$$

Hence,

$$\lim_{t,s\to t_0} \frac{1}{t-s}\log\delta_k\big(\Gamma^{-\frac{1}{2}}\Phi(t,s)^\top\Gamma\Phi(t,s)\Gamma^{-\frac{1}{2}}\big) = 2\delta_k\big(\Gamma^{-\frac{1}{2}} S_\Gamma(t_0)\Gamma^{-\frac{1}{2}}\big),$$

which completes the proof. □

Theorem 15 *(Relation to Γ-strain tensor). Let $\mathbb{R}^d$ be endowed with $\|\cdot\|_\Gamma$-norm, where Γ is a symmetric positive definite matrix in $\mathbb{R}^{d\times d}$. The following statements hold*:

(i) *Let J be a compact subset of I and $t\in I$ an accumulation point of J. Then, for all $k\in\{1,\dots,d\}$ we have*

$$\underline{\lambda}^{(k)}(J)\leqslant\delta_k\big(\Gamma^{-\frac{1}{2}} S_\Gamma(t)\Gamma^{-\frac{1}{2}}\big),\qquad \bar{\lambda}^{(k)}(J)\geqslant\delta_{d-k}\big(\Gamma^{-\frac{1}{2}} S_\Gamma(t)\Gamma^{-\frac{1}{2}}\big).$$

(ii) *Let $(J_m)_{m\in\mathbb{N}}$ be a sequence of compact subsets of I with $|J_m|\geqslant 2$ for each $m\in\mathbb{N}$ which converges to $t_0\in I$, i.e. $\lim_{m\to\infty} d_H(J_m,\{t_0\})=0$. Then in the sense of Hausdorff distance one has*

$$\lim_{m\to\infty}\Sigma_A(J_m)=\big\{\delta_1\big(\Gamma^{-\frac{1}{2}} S_\Gamma(t_0)\Gamma^{-\frac{1}{2}}\big),\dots,\delta_d\big(\Gamma^{-\frac{1}{2}} S_\Gamma(t_0)\Gamma^{-\frac{1}{2}}\big)\big\}.$$

Proof. (i) Define $t_1=\min J$. Since t is a limit point of J, there exists a sequence $(t_n)_{n\in\mathbb{N}}\subset J$ such that

$$\lim_{n\to\infty} t_n = t \quad\text{and}\quad |t_{n+1}-t|\leqslant|t_n-t|\quad\text{for } n\in\mathbb{N}.$$

Fix $k \in \{1, \dots, d\}$. From Definition 7, we derive that

$$\underline{\lambda}^{(k)}(J) \leqslant \underline{\lambda}^{(k)}\big(\{t_n, t_{n+1}\}\big) \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

By virtue of Remark 6 and Definition 7, we have

$$\underline{\lambda}^{(k)}\big(\{t_n, t_{n+1}\}\big) = \sup_{X \in G_{k,d}} \inf_{\xi \in X \setminus \{0\}} \frac{1}{t_{n+1} - t_n} \log \frac{\|\Phi(t_{n+1}, t_1)\xi\|_\Gamma}{\|\Phi(t_n, t_1)\xi\|_\Gamma}.$$

To simplify the notation, define $M := \Gamma^{-\frac{1}{2}} \Phi(t_{n+1}, t_n)^\top \Gamma \Phi(t_{n+1}, t_n) \Gamma^{-\frac{1}{2}}$. Let $X \in G_{k,d}$ be an arbitrary subspace of dimension k and $\xi_1, \dots, \xi_d$ denote the eigenvectors of M corresponding to the eigenvalues $\delta_1(M), \dots, \delta_d(M)$, respectively. Since $\dim X = k$ it follows that there exists a non-zero vector ξ such that $\Gamma^{\frac{1}{2}} \xi \in \Phi(t_n, t_1)^{-1} X \cap \operatorname{span}\{\xi_k, \dots, \xi_d\}$. Thus, $\Gamma^{\frac{1}{2}} \xi$ is of the form $\Gamma^{\frac{1}{2}} \xi = \alpha_k \xi_k + \dots + \alpha_d \xi_d$ with $\alpha_k^2 + \dots + \alpha_d^2 \neq 0$. Consequently,

$$\begin{aligned}
\underline{\lambda}^{(k)}\big(\{t_n, t_{n+1}\}, X\big) &\leqslant \frac{1}{2(t_{n+1} - t_n)} \log \frac{\|\Phi(t_{n+1}, t_n)\xi\|_\Gamma^2}{\|\xi\|_\Gamma^2} \\
&= \frac{1}{2(t_{n+1} - t_n)} \log \frac{\sum_{i=k}^d \delta_k(M) \alpha_k^2 \|\xi_i\|_2^2}{\sum_{i=k}^d \alpha_i^2 \|\xi_i\|_2^2} \\
&\leqslant \frac{1}{2(t_{n+1} - t_n)} \log \delta_k(M),
\end{aligned}$$

which together with Lemma 14 and (13) proves that

$$\underline{\lambda}^{(k)}(J) \leqslant \delta_k\big(\Gamma^{-\frac{1}{2}} S_\Gamma(t) \Gamma^{-\frac{1}{2}}\big).$$

Similarly, we have $\bar{\lambda}^{(k)}(J) \geqslant \delta_{d-k}(\Gamma^{-\frac{1}{2}} S_\Gamma(t) \Gamma^{-\frac{1}{2}})$ and the proof of part (i) is complete.

(ii) According to Theorem 10, it is sufficient to show that

$$\lim_{m \to \infty} \underline{\lambda}^{(k)}(J_m) = \delta_k\big(\Gamma^{-\frac{1}{2}} S_\Gamma(t_0) \Gamma^{-\frac{1}{2}}\big). \tag{14}$$

For this purpose, we choose $X = \Gamma^{-\frac{1}{2}} \operatorname{span}\{\xi_1, \dots, \xi_k\}$, where $\xi_1, \dots, \xi_d$ are eigenvectors of the matrix $\Gamma^{-\frac{1}{2}} S_\Gamma(t_0) \Gamma^{-\frac{1}{2}}$ corresponding to eigenvalues $\delta_1(\Gamma^{-\frac{1}{2}} S_\Gamma(t_0) \Gamma^{-\frac{1}{2}}), \dots, \delta_d(\Gamma^{-\frac{1}{2}} S_\Gamma(t_0) \Gamma^{-\frac{1}{2}})$, respectively. According to Lemma 14(i), for all $\xi \in X \setminus \{0\}$ we obtain

$$\begin{aligned}
\lim_{t,s \to t_0} \frac{1}{t - s} \log \frac{\|\Phi(t, s)\xi\|_\Gamma}{\|\xi\|_\Gamma} = \frac{\langle \xi, S_\Gamma(t_0)\xi \rangle}{\|\xi\|_\Gamma^2} &= \frac{\langle \Gamma^{\frac{1}{2}} \xi, \Gamma^{-\frac{1}{2}} S_\Gamma(t_0) \Gamma^{-\frac{1}{2}} \Gamma^{\frac{1}{2}} \xi \rangle}{\langle \Gamma^{\frac{1}{2}} \xi, \Gamma^{\frac{1}{2}} \xi \rangle} \\
&\geqslant \delta_k\big(\Gamma^{-\frac{1}{2}} S_\Gamma(t_0) \Gamma^{-\frac{1}{2}}\big),
\end{aligned}$$

this implies together with the fact that $\lim_{m \to \infty} d_H(J_m, \{t_0\}) = 0$ that there exists $N \in \mathbb{N}$ such that for all $m \geqslant N$ the following inequality

$$\frac{1}{t - s} \log \frac{\|\Phi(t, s)\xi\|_\Gamma}{\|\xi\|_\Gamma} \geqslant \delta_k\big(\Gamma^{-\frac{1}{2}} S_\Gamma(t_0) \Gamma^{-\frac{1}{2}}\big) - \varepsilon$$

holds for all $t, s \in J_m$ and $\xi \in X \setminus \{0\}$. This together with Definition 5 gives that

$$\underline{\lambda}^{(k)}(J_m) \geqslant \delta_k\big(\Gamma^{-\frac{1}{2}} S_\Gamma(t_0)\Gamma^{-\frac{1}{2}}\big) - \varepsilon \quad \text{for all } m \geqslant N.$$

This together with part (i) proves (14) and the proof is complete. □

3.3. *Approximation of the finite-time spectrum*

In view of Theorem 10 to compute the finite-time spectrum of a given nonautonomous differential equation we need to compute the extremal growth rates associated with this system. The rest of this section is devoted to provide a general approximation scheme to compute these extremal growth rates. For this purpose, we need to prove the following things:

1. The extremal growth rates of the discretization converge to the ones of the given system (for converging time-sets), see Theorem 17.
2. The extremal growth rates of the approximation of the evolution operator converge to the exact extremal growth rates (for converging approximation error), see Theorem 19.

Before we state and prove the convergence results listed above, we discuss in the following remark an explicit bound of the finite-time spectrum.

Remark 16. For all $t, s \in I$ using the variation of constant formula $\Phi(t,s)\xi = \xi + \int_s^t A(u)\Phi(u,s)\xi \,\mathrm{d}u$, we obtain $\|\Phi(t,s)\xi\| \leqslant \|\xi\| + a\int_s^t \|\Phi(u,s)\xi\|\,\mathrm{d}u$, where $a := \max_{t\in I}\|A(t)\|$. Using Gronwall's inequality, we get that for all $t, s \in I$

$$\big\|\Phi(t,s)\xi\big\| \leqslant e^{a|t-s|}\|\xi\|, \quad \xi \in \mathbb{R}^d,$$

which also implies that for all $t, s \in I$

$$\big\|\Phi(t,s)\xi\big\| \geqslant e^{-a|t-s|}\|\xi\|, \quad \xi \in \mathbb{R}^d.$$

Therefore, $\Sigma_J(A) \subseteq [-a, a]$ for all compact subsets $J \subseteq I$.

Theorem 17. *Let J be a compact subset of I. Let $(J_m)_{m\in\mathbb{N}}$ be a sequence of compact subsets of J satisfying $\lim_{m\to\infty} d_H(J_m, J) = 0$. Then, for all $j \in \{1, \ldots, d\}$ we have*

$$\lim_{m\to\infty} \underline{\lambda}^{(j)}(J_m) = \underline{\lambda}^{(j)}(J) \quad \textit{and} \quad \lim_{m\to\infty} \overline{\lambda}^{(j)}(J_m) = \overline{\lambda}^{(j)}(J). \tag{15}$$

Consequently,

$$\lim_{m\to\infty} d_H\big(\Sigma_{J_m}(A), \Sigma_J(A)\big) = 0.$$

Proof. Define

$$t_1 := \min J, \quad \text{and} \quad t_1^{(m)} := \min J_m \quad \text{for all } m \in \mathbb{N}.$$

From $\lim_{m\to\infty} d_H(J_m, J) = 0$, we derive that $\lim_{m\to\infty} t_1^{(m)} = t_1$. For each $m \in \mathbb{N}$, since $J_m \subset J$ it follows together with Definitions 5 and 7 that

$$\underline{\lambda}^{(j)}(J_m) \geqslant \underline{\lambda}^{(j)}(J) \quad \text{and} \quad \overline{\lambda}^{(j)}(J_m) \leqslant \overline{\lambda}^{(j)}(J).$$

Hence, to obtain (15) it is sufficient to prove that for all $j \in \{1, 2, \dots, d\}$

$$\limsup_{m\to\infty} \underline{\lambda}^{(j)}(J_m) \leqslant \underline{\lambda}^{(j)}(J) \quad \text{and} \quad \liminf_{m\to\infty} \overline{\lambda}^{(j)}(J_m) \geqslant \overline{\lambda}^{(j)}(J). \tag{16}$$

For this purpose, let $(X_m)_{m\in\mathbb{N}}$ be a sequence in $G_{j,d}$ satisfying $\underline{\lambda}(J_m, X_m) = \underline{\lambda}^{(j)}(J_m) =: \alpha_m$. Suppose there exists $\delta > 0$ such that $\alpha := \underline{\lambda}^{(j)}(J) < \limsup_{m\to\infty} \alpha_m - \delta$. Taking subsequences, if necessary, we assume that $\lim_{m\to\infty} X_m = X \in G_{j,d}$ and $\lim_{m\to\infty} \alpha_m > \alpha + \delta$. Choose arbitrary $s, t \in J$ with $s < t$ and $\xi \in X \setminus \{0\}$. By $\lim_{m\to\infty} d_H(J_m, J) = 0$ and $\lim_{m\to\infty} d_G(X, X_m) = 0$, there exist sequences $(s_m)_{m\in\mathbb{N}}, (t_m)_{m\in\mathbb{N}}, (\xi_m)_{m\in\mathbb{N}}$ satisfying $s_m, t_m \in J_m$ with $s_m \leqslant t_m$, $\xi_m \in X_m$ and

$$\lim_{m\to\infty} s_m = s, \qquad \lim_{m\to\infty} t_m = t, \qquad \lim_{m\to\infty} \xi_m = \xi.$$

By Definition 5, we get for all $m \in \mathbb{N}$

$$\left\| \Phi\left(t_m, t_1^{(m)}\right)\xi_m \right\| \geqslant e^{\alpha_m(t_m - s_m)} \left\| \Phi\left(s_m, t_1^{(m)}\right)\xi_m \right\|.$$

In the limit for $m \to \infty$ we get

$$\left\| \Phi(t, t_1)\xi \right\| \geqslant e^{(\alpha+\delta)(t-s)} \left\| \Phi(s, t_1)\xi \right\|,$$

which implies together with Definition 5 that $\underline{\lambda}(J, X) \geqslant \alpha + \delta$ and contradicts to Definition 7. Similarly, we have $\liminf_{m\to\infty} \overline{\lambda}^{(j)}(J_m) \geqslant \overline{\lambda}^{(j)}(J)$ for each $j \in \{1, \dots, d\}$. So, (16) is proved and the proof is complete. $\square$

We now provide an example to illustrate that the condition that J_m is a subset of J for all $m \in \mathbb{N}$ is necessary in Theorem 17.

Example 18. Consider the scalar differential equation on $[0, 1]$

$$\dot{x}(t) = a(t)x(t),$$

where $a : [0, 1] \to \mathbb{R}$ is a continuous function. Set $J := \{0, 1\}$ and $J_m := \{0, 1 - \frac{1}{m}, 1\}$. Clearly, $\lim_{m\to\infty} d_H(J_m, J) = 0$. An elementary computation yields that

$$\Sigma_J(a) = \left\{ \int_0^1 a(t)\,dt \right\},$$

$$\lim_{m\to\infty} \Sigma_{J_m}(a) = \left[\min\left\{ \int_0^1 a(t)\,dt, a(1) \right\}, \max\left\{ \int_0^1 a(t)\,dt, a(1) \right\} \right].$$

Thus, if $\int_0^1 a(t)\,dt \neq a(1)$ then $\lim_{m\to\infty} d_H(\Sigma_J, \Sigma_{J_m}) \neq 0$. An explicit function a satisfying this condition is $a = \mathrm{id}_{[0,1]}$.

Theorem 19. *Consider system* (1) *on a compact interval $I \subseteq \mathbb{R}$. Let $J = \{t_1, \dots, t_k\}$ with $t_1 < t_2 < \dots < t_k$ be a subset of I. Set*

$$a := \max_{t\in I} \left\| A(t) \right\|, \qquad \Delta := \min_{i\in\{1,\dots,k-1\}} t_{i+1} - t_i, \qquad T := t_+ - t_-.$$

Let $\Psi_2, \ldots, \Psi_k \in \mathbb{R}^{d\times d}$ be invertible matrices, set $\Psi_1 = \mathrm{id} \in \mathbb{R}^{d\times d}$. For $j \in \{1, \ldots, d\}$, we define

$$\underline{\lambda}^{(j)}_{\mathrm{approx}}(J) := \sup_{X\in G_{j,d}} \inf_{\xi\in X\cap S^{d-1}} \min_{i\in\{1,\ldots,k-1\}} \frac{1}{t_{i+1}-t_i} \log \frac{\|\Psi_{i+1}\xi\|}{\|\Psi_i\xi\|} \tag{17}$$

and

$$\overline{\lambda}^{(j)}_{\mathrm{approx}}(J) := \inf_{X\in G_{j,d}} \sup_{\xi\in X\cap S^{d-1}} \max_{i\in\{1,\ldots,k-1\}} \frac{1}{t_{i+1}-t_i} \log \frac{\|\Psi_{i+1}\xi\|}{\|\Psi_i\xi\|}. \tag{18}$$

Let $\varepsilon > 0$ and $\delta < \varepsilon \mathrm{e}^{-aT} \min\{\frac{1}{2}, \frac{\Delta}{3}\}$. Then

$$\big\|\Psi_n - \Phi(t_n, t_1)\big\| < \delta \quad \textit{for each } n \in \{2, \ldots, k\} \tag{19}$$

implies

$$\big|\underline{\lambda}^{(j)}_{\mathrm{approx}}(J) - \underline{\lambda}^{(j)}(J)\big| < \varepsilon \quad \textit{and} \quad \big|\overline{\lambda}^{(j)}_{\mathrm{approx}}(J) - \overline{\lambda}^{(j)}(J)\big| < \varepsilon. \tag{20}$$

Proof. Let $\xi \in S^{d-1}$, $i \in \{1, \ldots, k-1\}$ and $\varepsilon > 0$. Using (19), we obtain that

$$\begin{aligned}
\log \frac{\|\Phi(t_{i+1}, t_1)\xi\|}{\|\Phi(t_i, t_1)\xi\|} - \log \frac{\|\Psi_{i+1}\xi\|}{\|\Psi_i\xi\|} &= \log \frac{\|\Phi(t_{i+1}, t_1)\xi\|}{\|\Psi_{i+1}\xi\|} - \log \frac{\|\Psi_i\xi\|}{\|\Phi(t_i, t_1)\xi\|} \\
&\leqslant \log \frac{\|\Phi(t_{i+1}, t_1)\xi\|}{\|\Phi(t_{i+1}, t_1)\xi\| - \delta} + \log \frac{\|\Phi(t_i, t_1)\xi\| + \delta}{\|\Phi(t_i, t_1)\xi\|}.
\end{aligned}$$

By virtue of Remark 16, we get that $\|\Phi(t_i, t_1)\xi\| \geqslant \mathrm{e}^{-a(t_i - t_1)} \geqslant \mathrm{e}^{-aT}$. Hence,

$$\log \frac{\|\Phi(t_{i+1}, t_1)\xi\|}{\|\Phi(t_i, t_1)\xi\|} - \log \frac{\|\Psi_{i+1}\xi\|}{\|\Psi_i\xi\|} \leqslant \log\left(1 + \frac{\delta}{\mathrm{e}^{-aT} - \delta}\right) + \log\left(1 + \frac{\delta}{\mathrm{e}^{-aT}}\right).$$

Consequently, for all $j \in \{1, \ldots, d\}$ we get

$$\underline{\lambda}^{(j)}(J) - \underline{\lambda}^{(j)}_{\mathrm{approx}} \leqslant \frac{1}{\Delta}\left[\log\left(1 + \frac{\delta}{\mathrm{e}^{-aT} - \delta}\right) + \log\left(1 + \frac{\delta}{\mathrm{e}^{-aT}}\right)\right].$$

By using the obvious inequality $\log(1 + x) \leqslant x$ for all $x \in [0, \infty)$, we obtain

$$\underline{\lambda}^{(j)}(J) - \underline{\lambda}^{(j)}_{\mathrm{approx}} \leqslant \frac{\delta \mathrm{e}^{aT}}{\Delta}\left(\frac{1}{1 - \delta \mathrm{e}^{aT}} + 1\right) < \frac{3\mathrm{e}^{aT}}{\Delta}\delta < \varepsilon,$$

for $\delta < \varepsilon \mathrm{e}^{-aT} \min\{\frac{1}{2}, \frac{\Delta}{3}\}$. The above argument applies equally well when interchanging Φ and Ψ, such that we get

$$\big|\underline{\lambda}^{(j)}(J) - \underline{\lambda}^{(j)}_{\mathrm{approx}}\big| < \varepsilon$$

in case of (19) with $\delta < \varepsilon \mathrm{e}^{-aT} \min\{\frac{1}{2}, \frac{\Delta}{3}\}$, which proves the first part of (20). Analogously, we get the second part of (20) and the proof is complete. □

By combining Theorems 17 and 19, we are now in a position to state our final approximation result for the extremal growth rates.

Theorem 20. *Consider system* (1) *on a compact interval $I \subseteq \mathbb{R}$, let $J \subseteq I$ be compact, $\varepsilon > 0$ and set $a := \max_{t\in I} \|A(t)\|$ and $T := t_+ - t_-$. Let $(J_m)_{m\in\mathbb{N}}$ be a sequence of finite subsets of J with $\lim_{m\to\infty} d_H(I, J_m) = 0$, where $|J_m| = k_m$, $J_m = \{t_-, t_{m,2}, \dots, t_{m,k_m}\}$, $\Delta_m := \rho(J_m)$ and $\varepsilon_m := \frac{\Delta_m}{3e^{aT}}\varepsilon$. For each $m \in \mathbb{N}$ let $\Psi_{m,2}, \dots, \Psi_{m,k_m}$ be ε_m-approximations of $\Phi(t_{m,2}, t_-), \dots, \Phi(t_{m,k_m}, t_-)$ in the sense of* (19)*. Then the following limit inequalities hold*:

$$\left|\underline{\lambda}^{(j)}(J) - \lim_{m\to\infty} \underline{\lambda}^{(j)}_{\text{approx}}(J_m)\right|, \left|\bar{\lambda}^{(j)}(J) - \lim_{m\to\infty} \bar{\lambda}^{(j)}_{\text{approx}}(J_m)\right| \leqslant \varepsilon.$$

Theorem 20 provides an approximation result for the spectral growth rates $\underline{\lambda}^{(j)}(J)$ and $\bar{\lambda}^{(j)}(J)$ if J is approximated by J_m and the evolution operator Φ is only available approximately. For a numerical implementation one needs to also address the question of computing the growth rates $\underline{\lambda}^{(j)}_{\text{approx}}(J_m)$ and $\bar{\lambda}^{(j)}_{\text{approx}}(J_m)$. The following remark provides the answer of this question for planar systems which are endowed with the Euclidean norm $\|\cdot\|_2 = \|\cdot\|_{\text{id}}$ and J_m consists of equidistant grid points.

Remark 21. Consider Theorem 19 for $d = 2$ with approximating matrices $\Psi_2, \dots, \Psi_k \in \mathbb{R}^{2\times 2}$ as in (19) and the points in $J = \{t_1, \dots, t_k\}$ are equidistant. To compute $\underline{\lambda}^{(j)}_{\text{approx}}(J)$ and $\bar{\lambda}^{(j)}_{\text{approx}}(J)$ using formulas (17) and (18), we first observe that if there exist $i, j \in \{1, \dots, k-1\}$, $i \neq j$, such that for all $\xi \in \mathbb{R}^2 \setminus \{0\}$ the equality $\frac{1}{t_{i+1}-t_i} \log \frac{\|\Psi_{i+1}\xi\|_2}{\|\Psi_i\xi\|_2} = \frac{1}{t_{j+1}-t_j} \log \frac{\|\Psi_{j+1}\xi\|_2}{\|\Psi_j\xi\|_2}$ holds, then one can w.l.o.g. remove either i or j from $\{1, \dots, k-1\}$ without changing the value of (17) and (18). Hence we can assume that for each $i, j \in \{1, \dots, k-1\}$, $i \neq j$, there exists $\xi \in \mathbb{R}^2 \setminus \{0\}$ such that

$$\frac{1}{t_{i+1}-t_i} \log \frac{\|\Psi_{i+1}\xi\|_2}{\|\Psi_i\xi\|_2} \neq \frac{1}{t_{j+1}-t_j} \log \frac{\|\Psi_{j+1}\xi\|_2}{\|\Psi_j\xi\|_2}.$$

Under this assumption, the equation $\log \frac{\|\Psi_{i+1}\xi\|_2}{\|\Psi_i\xi\|_2} = \log \frac{\|\Psi_{j+1}\xi\|_2}{\|\Psi_j\xi\|_2}$ can be solved as

$$\|\Psi_{i+1}\xi\|_2^2 \|\Psi_j\xi\|_2^2 - \|\Psi_{j+1}\xi\|_2^2 \|\Psi_i\xi\|_2^2 = 0.$$

As a consequence, the set

$$\mathcal{M} := \bigcup_{i,j=1,\, i\neq j}^{k-1} \left\{\xi \in S^1\colon \frac{1}{t_{i+1}-t_i} \log \frac{\|\Psi_{i+1}\xi\|_2}{\|\Psi_i\xi\|_2} = \frac{1}{t_{j+1}-t_j} \log \frac{\|\Psi_{j+1}\xi\|_2}{\|\Psi_j\xi\|_2}\right\}$$

contains at most $4(k-1)(k-2)$ elements. Define

$$\underline{\lambda} := \max_{\xi\in\mathcal{M}} \min_{i\in\{1,\dots,k-1\}} \frac{1}{t_{i+1}-t_i} \log \frac{\|\Psi_{i+1}\xi\|}{\|\Psi_i\xi\|},$$
$$\bar{\lambda} := \min_{\xi\in\mathcal{M}} \max_{i\in\{1,\dots,k-1\}} \frac{1}{t_{i+1}-t_i} \log \frac{\|\Psi_{i+1}\xi\|}{\|\Psi_i\xi\|}.$$

Let

$$J^* := \bigl\{j \in \{1, \dots, k\}\colon \delta_1\bigl(\Psi_{j+1}\Psi_j^{-1}\bigr) \neq \delta_2\bigl(\Psi_{j+1}\Psi_j^{-1}\bigr)\bigr\}.$$

For $j \in J^*$, define $\xi_{1,j}, \xi_{2,j} \in \mathbb{R}^2$ with $\|\xi_{1,j}\| = \|\xi_{2,j}\| = 1$ such that

$$\bigl\|\Psi_{j+1}\Psi_j^{-1}\xi_{1,j}\bigr\|_2 = \delta_1\bigl(\Psi_{j+1}\Psi_j^{-1}\bigr), \qquad \bigl\|\Psi_{j+1}\Psi_j^{-1}\xi_{2,j}\bigr\|_2 = \delta_2\bigl(\Psi_{j+1}\Psi_j^{-1}\bigr).$$

Using the fact that

$$\sup_{\xi \in S^1} \frac{\|A\xi\|_2}{\|\xi\|_2} = \delta_1(A),$$

we observe that for $j \in J^*$ the unique vector ξ with $\|\Psi_j \xi\|_2 = 1$ which solves the optimization problem

$$\max_{\xi \in S^1} \frac{1}{t_{j+1} - t_j} \frac{\|\Psi_{j+1}\xi\|_2}{\|\Psi_j \xi\|_2}$$

is given by $\xi = \Psi_j^{-1}\xi_{1,j}$. Analogously, the unique vector ξ with $\|\Psi_j \xi\|_2 = 1$ which realizes the optimization problem

$$\min_{\xi \in S^1} \frac{1}{t_{j+1} - t_j} \frac{\|\Psi_{j+1}\xi\|_2}{\|\Psi_j \xi\|_2}$$

is given by $\xi = \Psi_j^{-1}\xi_{2,j}$. These facts, together with a simple but technical computation, lead to the following formulas to compute $\underline{\lambda}^{(j)}_{\text{approx}}(J)$ and $\bar{\lambda}^{(j)}_{\text{approx}}(J)$:

- Computation of $\underline{\lambda}^{(1)}_{\text{approx}}(J)$:

$$\underline{\lambda}^{(1)}_{\text{approx}}(J) = \max\left\{\underline{\lambda}, \max_{j \in J^*} \min_{i=1,\dots,k-1} \frac{1}{t_{i+1} - t_i} \log \frac{\|\Psi_{i+1}\Psi_j^{-1}\xi_{1,j}\|_2}{\|\Psi_i \Psi_j^{-1}\xi_{1,j}\|_2}\right\}.$$

- Computation of $\bar{\lambda}^{(1)}_{\text{approx}}(J)$:

$$\bar{\lambda}^{(1)}_{\text{approx}}(J) = \min\left\{\bar{\lambda}, \min_{j \in J^*} \max_{i=1,\dots,k-1} \frac{1}{t_{i+1} - t_i} \log \frac{\|\Psi_{i+1}\Psi_j^{-1}\xi_{2,j}\|_2}{\|\Psi_i \Psi_j^{-1}\xi_{2,j}\|_2}\right\}.$$

- Computation of $\underline{\lambda}^{(2)}_{\text{approx}}(J)$:

$$\underline{\lambda}^{(2)}_{\text{approx}}(J) = \min_{i=1,\dots,k-1} \frac{1}{t_{i+1} - t_i} \log \delta_2\left(\Psi_{i+1}\Psi_i^{-1}\right).$$

- Computation of $\bar{\lambda}^{(2)}_{\text{approx}}(J)$:

$$\bar{\lambda}^{(2)}_{\text{approx}}(J) = \max_{i=1,\dots,k-1} \frac{1}{t_{i+1} - t_i} \log \delta_1\left(\Psi_{i+1}\Psi_i^{-1}\right).$$

4. Finite-time Lyapunov spectrum

In the previous section, we established a general theory for finite-time spectrum with respect to an arbitrary norm and an arbitrary compact time-set. Now we are dealing with the special case that the state space is endowed with the Euclidean norm $\|\cdot\|_2 = \|\cdot\|_{\text{id}}$ and the compact subset $J \subset I$ contains only two time-points, i.e. $J = \{t_-, t_+\}$. Our aim is to show that the finite-time spectrum in this special case coincides with the finite-time Lyapunov spectrum. We refer the reader to [19] for an illustration of the importance of finite-time Lyapunov exponents in detecting Lagrangian coherent structures on finite-time intervals, e.g. in fluid dynamics and oceanography, cf. also [24,15] and the references therein.

We do not intend to recall the general theory of classical Lyapunov exponents for nonautonomous linear differential equations, but to motivate the definition of finite-time Lyapunov spectrum, let us have a rough look at this classical concept for certain nonautonomous linear differential equations (see e.g. [1,18,22] for more details on so-called regular systems). Consider system (1) on the unbounded interval $I = [0, \infty)$. Let the following $d \times d$ matrix

$$\Lambda := \lim_{t\to\infty} \left(\Phi(t,0)^\top \Phi(t,0)\right)^{\frac{1}{2t}}$$

exist. Then the classical *Lyapunov exponents* are the logarithms of the eigenvalues of Λ. The set of all Lyapunov exponents is called *Lyapunov spectrum*.

As a possible adaptation of the concept of Lyapunov exponents to ODEs (1) on a compact time-set $I = [t_-, t_+]$, the *finite-time Lyapunov exponents* can be defined as the numbers in the set

$$\Sigma_{\mathrm{FTLE},[t_-,t_+]}(A) := \left\{ \frac{1}{t_+ - t_-} \log\sqrt{\lambda}\colon\ \lambda \in \sigma\left(\Phi(t_+,t_-)^\top \Phi(t_+,t_-)\right) \right\}, \tag{21}$$

which we call the *finite-time Lyapunov spectrum* of (1) on the interval I. For precursors of the finite-time Lyapunov spectrum $\Sigma_{\mathrm{FTLE},[t_-,t_+]}(A)$ see e.g. [20,16] for the largest and the smallest finite-time Lyapunov exponent, formulas (8) and (9) in [15] and the numerical approach in [17].

Remark 22. By the fact that eigenvalues depend continuously on the entries of matrices, we obtain that for $t_+ \to \infty$ the finite-time Lyapunov spectrum $\Sigma_{\mathrm{FTLE},[t_-,t_+]}(A)$ tends to the classical Lyapunov spectrum provided that the limit $\lim_{t\to\infty}(\Phi(t,0)^\top\Phi(t,0))^{\frac{1}{2t}}$ exists.

An alternative approach to Lyapunov spectrum is based on the fact that the maximal growth rate of a k-dimensional infinitesimal volume is the k-th Lyapunov exponent (see e.g. [2,25]). In this section, we prove this analog relation between finite-time maximal growth rate and the finite-time Lyapunov spectrum defined as in (21). To formulate and prove this result, we consider system (1) on an interval $I \supseteq [t_-, t_+]$ and compute explicitly the maximal and minimal growth rates with respect to $J = \{t_-, t_+\}$.

Proposition 23. *Consider system* (1) *on the interval I. Suppose that* $\mathbb{R}^d$ *is endowed with the standard Euclidean norm and* $J = \{t_-, t_+\} \subset I$. *Let* $\delta_n < \cdots < \delta_1$ *be the singular values of* $\Phi(t_+, t_-)$ *and* d_j *the multiplicities of* δ_j, $j \in \{1, \dots, n\}$. *Then for all* $j \in \{1, \dots, n\}$ *we have*

$$\underline{\lambda}^{(k)}(J) = \frac{1}{t_+ - t_-} \log\delta_j \quad \textit{for all} \sum_{\ell=1}^{j-1} d_\ell < k \leqslant \sum_{\ell=1}^{j} d_\ell, \tag{22}$$

and

$$\bar{\lambda}^{(k)}(J) = \frac{1}{t_+ - t_-} \log\delta_j \quad \textit{for all} \sum_{\ell=j}^{n} d_\ell \geqslant k > \sum_{\ell=j+1}^{n} d_\ell. \tag{23}$$

Proof. Let $\Phi(t_+, t_-) = VDU$ be a singular value decomposition, i.e. U, V are orthogonal matrices in $\mathbb{R}^{d\times d}$ and

$$D = \mathrm{diag}(\delta_1, \dots, \delta_1, \delta_2, \dots, \delta_2, \dots, \delta_m, \dots, \delta_m).$$

Let $k \in \{1, \dots, d\}$ and $j \in \{1, \dots, m\}$ satisfy that $\sum_{\ell=1}^{j-1} d_\ell < k \leqslant \sum_{\ell=1}^{j} d_\ell$. Our aim is to show that $\underline{\lambda}^{(k)}(J) = \frac{1}{t_+ - t_-} \log \delta_j$. For this purpose, we set $X := \operatorname{span}\{U^{-1}e_1, \dots, U^{-1}e_k\}$, where $\{e_1, \dots, e_d\}$ denotes the standard basis of $\mathbb{R}^d$. By virtue of Remark 6, we get

$$\underline{\lambda}(J, X) = \frac{1}{t_+ - t_-} \inf_{\xi \in X \cap \mathcal{S}^{d-1}} \log \left\| \Phi(t_+, t_-)\xi \right\| = \frac{1}{t_+ - t_-} \inf_{\xi \in X \cap \mathcal{S}^{d-1}} \log \| DU\xi \|.$$

For each $\xi \in X \cap \mathcal{S}^{d-1}$, we have $U\xi = \sum_{\ell=1}^{k} \alpha_\ell e_\ell$ for some $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ and $\sum_{i=1}^{k} \alpha_i^2 = 1$. Hence, we have

$$\begin{aligned} \log \| DU\xi \| &= \log \sqrt{\delta_1^2 \sum_{i=1}^{d_1} \alpha_i^2 + \delta_2^2 \sum_{i=d_1+1}^{d_1+d_2} \alpha_i^2 + \dots + \delta_j^2 \sum_{i=d_1+\dots+d_{j-1}+1}^{k} \alpha_i^2} \\ &\geqslant \log \delta_j. \end{aligned}$$

Therefore, $\underline{\lambda}^{(k)}(J) \geqslant \underline{\lambda}(J, X) \geqslant \frac{1}{t_+ - t_-} \log \delta_j$. Now let $Y \in G_{k,d}$. Then, $M := Y \cap \operatorname{span}\{U^{-1}e_d, \dots, U^{-1}e_{d_1+\dots+d_{j-1}+1}\} \neq \{0\}$. We choose $\xi \in M \cap \mathcal{S}^{d-1}$ such that $U\xi = \sum_{i=d_1+\dots+d_{j-1}+1}^{d} \alpha_i e_i$ with $\sum_{i=d_1+\dots+d_{j-1}+1}^{d} \alpha_i^2 = 1$. Similar to the above estimate, we get $\log \| DU\xi \| \leqslant \log \delta_j$ and hence $\underline{\lambda}^{(k)}(J) \leqslant \frac{1}{t_+ - t_-} \log \delta_j$. So (22) is proved. Similarly, we also have (23) and the proof is complete. □

Now we are in a position to state and prove that the general notion of finite-time spectrum developed with respect to the set containing only the starting and ending times coincides with the finite-time Lyapunov spectrum. It is an improvement of the spectral theorem given in [21, Theorem 4.13].

Theorem 24. *Consider system* (1) *on an interval $I \subseteq \mathbb{R}$. Suppose that $\mathbb{R}^d$ is endowed with the standard Euclidean norm and $J = \{t_-, t_+\} \subset I$. Let $\delta_n < \dots < \delta_1$ denote the singular values of $\Phi(t_+, t_-)$. Then*

$$\Sigma_J(A) = \Sigma_{\mathrm{FTLE},[t_-,t_+]}(A) = \left\{ \frac{1}{t_+ - t_-} \log \delta_1, \dots, \frac{1}{t_+ - t_-} \log \delta_n \right\}.$$

Proof. Let $\Lambda := \{i_0, \dots, i_n\}$, $i_0 < \dots < i_n$, be the set of all indices $j \in \{0, \dots, d\}$ satisfying $\overline{\lambda}^{(j)}(J) < \underline{\lambda}^{(d-j)}(J)$. According to Proposition 23, we obtain that

$$\Lambda = \{0, d_n, d_n + d_{n-1}, \dots, d_n + \dots + d_1\},$$

which together with Theorem 10 implies that

$$\begin{aligned} \Sigma_J(A) &= \left[\underline{\lambda}^{(d)}, \overline{\lambda}^{(i_1)}\right] \cup \left[\underline{\lambda}^{(d-i_1)}, \overline{\lambda}^{(i_2)}\right] \cup \dots \cup \left[\underline{\lambda}^{(d-i_{n-1})}, \overline{\lambda}^{(i_n)}\right] \\ &= \left\{ \frac{1}{t_+ - t_-} \log \delta_1, \dots, \frac{1}{t_+ - t_-} \log \delta_n \right\}, \end{aligned}$$

where we omitted the dependency of the extremal growth rates on J for simplicity. On the other hand, using the fact that

$$\sigma\left(M^\top M\right) = \left\{ \delta^2\colon\ \delta \text{ is a singular value of } M \right\}$$

and (21) we have

$$\Sigma_{\mathrm{FTLE},[t_-,t_+]}(A) = \left\{ \frac{1}{t_+ - t_-} \log \delta_1, \ldots, \frac{1}{t_+ - t_-} \log \delta_n \right\},$$

which completes the proof. □

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References

[1] L.Ya. Adrianova, Introduction to Linear Systems of Differential Equations, Transl. Math. Monogr., American Mathematical Society, 1995.
[2] G. Benettin, L. Galgani, A. Giorgilli, J.M. Strelcyn, Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems. A method for computing all of them. Part 1: Theory, Meccanica 15 (1980) 9–20.
[3] A. Berger, On finite-time hyperbolicity, Commun. Pure Appl. Anal. 10 (2011) 963–981.
[4] A. Berger, T.S. Doan, S. Siegmund, Nonautonomous finite-time dynamics, Discrete Contin. Dyn. Syst. Ser. B 9 (3–4) (2008) 463–492.
[5] A. Berger, T.S. Doan, S. Siegmund, A remark on finite-time hyperbolicity, Proc. Appl. Math. Mech. 8 (2008) 10917–19918.
[6] A. Berger, T.S. Doan, S. Siegmund, A definition of spectrum for differential equations on finite time, J. Differential Equations 246 (3) (2009) 1098–1118.
[7] M. Berger, B. Gostiaux, Differential Geometry: Manifolds, Curves, and Surfaces, Grad. Texts in Math., vol. 115, Springer-Verlag, New York, 1988.
[8] T.S. Doan, K. Palmer, S. Siegmund, Transient spectral theory, stable and unstable cones and Gershgorin's theorem for finite-time differential equations, J. Differential Equations 250 (11) (2011) 4177–4199.
[9] L.H. Duc, S. Siegmund, Existence of finite-time hyperbolic trajectories for planar Hamiltonian flows, J. Dynam. Differential Equations 23 (3) (2011) 475–494.
[10] L.H. Duc, S. Siegmund, Hyperbolicity and invariant manifolds for planar nonautonomous systems on finite time intervals, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 18 (3) (2008) 641–674.
[11] G.H. Golub, C.-F. Van Loan, Matrix Computations, Johns Hopkins Stud. Math. Sci., The Johns Hopkins Univ. Press., Baltimore, MD, 1996.
[12] G. Haller, Lagrangian structures and the rate of strain in a partition of two-dimensional turbulence, Phys. Fluids 13 (2001) 3365–3385.
[13] G. Haller, Distinguished material surfaces and coherent structures in three-dimensional fluid flows, Phys. D 149 (4) (2001) 248–277.
[14] G. Haller, An objective definition of a vortex, J. Fluid Mech. 525 (2005) 1–26.
[15] G. Haller, A variational theory of hyperbolic Lagrangian coherent structures, Phys. D 240 (2011) 574–598.
[16] G. Haller, T. Sapsis, Lagrangian coherent structures and the smallest finite-time Lyapunov exponent, Chaos 21 (2011) 023115, 1–5.
[17] G.V. Kuptsov, U. Parlitz, Theory and computation of covariant Lyapunov vectors, arXiv:1105.5228v1, 26 May 2011.
[18] V.I. Oseledets, A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems, Trans. Moscow Math. Soc. 19 (1968) 197–231.
[19] T. Peacock, J. Dabiri, Introduction to focus issue: Lagrangian coherent structures, Chaos 20 (2010) 017501, 1.
[20] R.T. Pierrehumbert, Large-scale horizontal mixing in planetary atmospheres, Phys. Fluids A 3 (1991) 1250–1260.
[21] M. Rasmussen, Finite-time attractivity and bifurcation for nonautonomous differential equations, Differ. Equ. Dyn. Syst. 18 (1–2) (2010) 57–78.
[22] D. Ruelle, Ergodic theory of differentiable dynamical systems, Inst. Hautes Études Sci. Publ. Math. 50 (1979) 27–58.
[23] R.J. Sacker, G.R. Sell, A spectral theory for linear differential systems, J. Differential Equations 27 (1978) 320–358.
[24] S.C. Shadden, F. Lekien, J.E. Marsden, Definition and properties of Lagrangian coherent structures from finite-time Lyapunov exponents in two-dimensional aperiodic flows, Phys. D 212 (3–4) (2005) 271–304.
[25] S.V. Ershov, A.B. Potapov, On the concept of stationary Lyapunov basis, Phys. D (1998) 167–198.
[26] S. Siegmund, Dichotomy spectrum for nonautonomous differential equations, J. Dynam. Differential Equations 14 (1) (2002) 243–258.